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GENERATION OF WAVE AMPLITUDE AND ORBITAL PARTICLE VELOCITY FIELD OF A RANDOM SEA

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(6) GENERATION OF WAVE AMPLITUDE AND ORBITAL PARTICLE VELOCITY FIELD OF A RANDOM SEA

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ABSTRACT

In designing vehicles capable of weathering rough seas produced by intense, persevering storms, some method of generating a representation of the important effects of a random sea must be employed. The proposed technique models the wave amplitude as a function of horizontal location and time and evaluates the accompanying horizontal and vertical components of orbital particle velocity as a function of horizontal and vertical location and time. The underlying definition of a random sea is in terms of a Gaussian, stationary random process with a Pierson-Moskowitz spectral density function for the wave amplitude at some reference point. The method is suitable for computing probability distributions for the vehicle state variables whether or not the vehicle dynamics are governed by a linear differential equation. If the dynamics are linear, the actual Gaussian stationary probability distribution can be computed through solution of an algebraic system. For the nonlinear case, random number generation and numerical integration produce time averages to approximate ensemble averages in estimating any number of statistical moments.

GENERATION OF WAVE AMPLITUDE AND ORBITAL PARTICLE VELOCITY FIELD OF A RANDOM SEA

Over the past several years in activities related to hydrofoil design, Grumman engineers have amassed a wealth of experience in predicting the statistics of the response of vehicles to random seas. Most of this work has been in designing ships that use fully submerged hydrofoils for support (Ref. 1). Currently, this work is being applied to the design of a buoy. We have developed a unique mathematical representation of random seas that allows precise modeling of distributed effects on water-supported vehicles. This representation is far superior to the usual filtered white noise models that must rely on such artifices as average wave celerity to estimate phase shift as a function of location.

Our representation is an adaptation of Papoulis's approximate Fourier expansion (Ref. 2), which leads to a mean-square almost-periodic random process. We obtain a better wave representation by taking N equiprobable frequencies instead of N equally spaced ones. In fact, Gikhman and Skorokhod (Ref. 3) show that any such scheme of choosing frequencies leads to a convergent approximation to the random process as N increases without bound; but in practice, the equiprobable method converges faster as a function of N. Figure 1 represents a typical convergence rate of the truncation error in vehicle response statistics.

Perhaps the best way to describe the determination of the approximate random process is to develop graphically a fifth order representation. The actual 19-knot Pierson-Moskowitz spectral density (Ref. 4) representing a typical sea state 4 is depicted in Fig. 2a. Figure 2b is the corresponding spectral function (the integral of the spectral density in the notation of Gikhman and Skorokhod). The ordinate of the spectral function is divided into

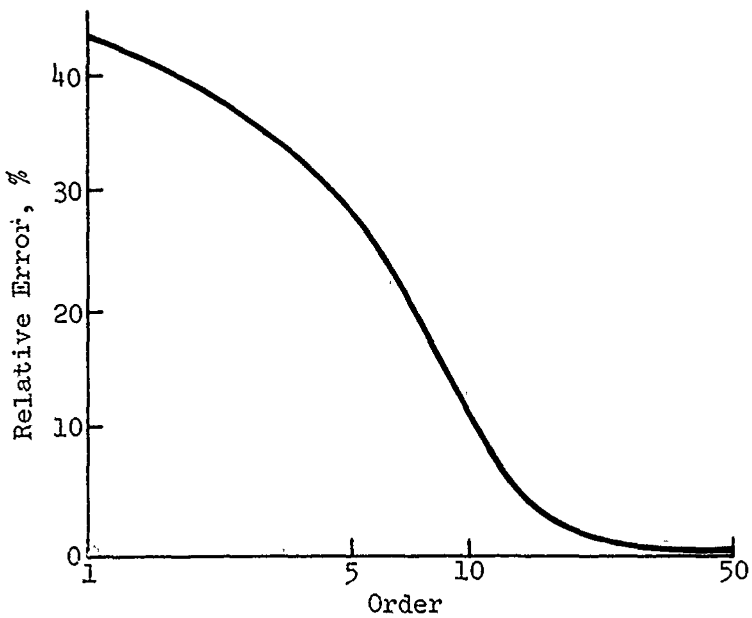

Fig. 1 Relative Error in the Response Statistics versus Order of the Approximation

five equal parts from its origin to its final value (which is the variance, σ^2, of the process). The center of each of these partitions along the ordinate determines five discrete frequencies, ω_i. (The dotted lines on the figure shows how these are determined.) The resulting approximate spectral function is a staircase with equal height and variable width steps. To the left of each step the given spectral function is underestimated, while to the right it is overestimated by the same amount. The approximate spectral density is a train of equal strength impulses at ω_1, ω_2, ω_3, ω_4, and ω_5. Note that this method concentrates the frequencies near the peak of the spectral density. The actual determination of the ω_i's for any N has been computerized using the Newton-Raphson algorithm.

Once the variance, σ^2, the order, N, and the frequencies, ω_i, $i = 1, 2, \ldots, N$, are specified, the approximate random process for wave amplitude, η, as a function of time, t, takes the form:

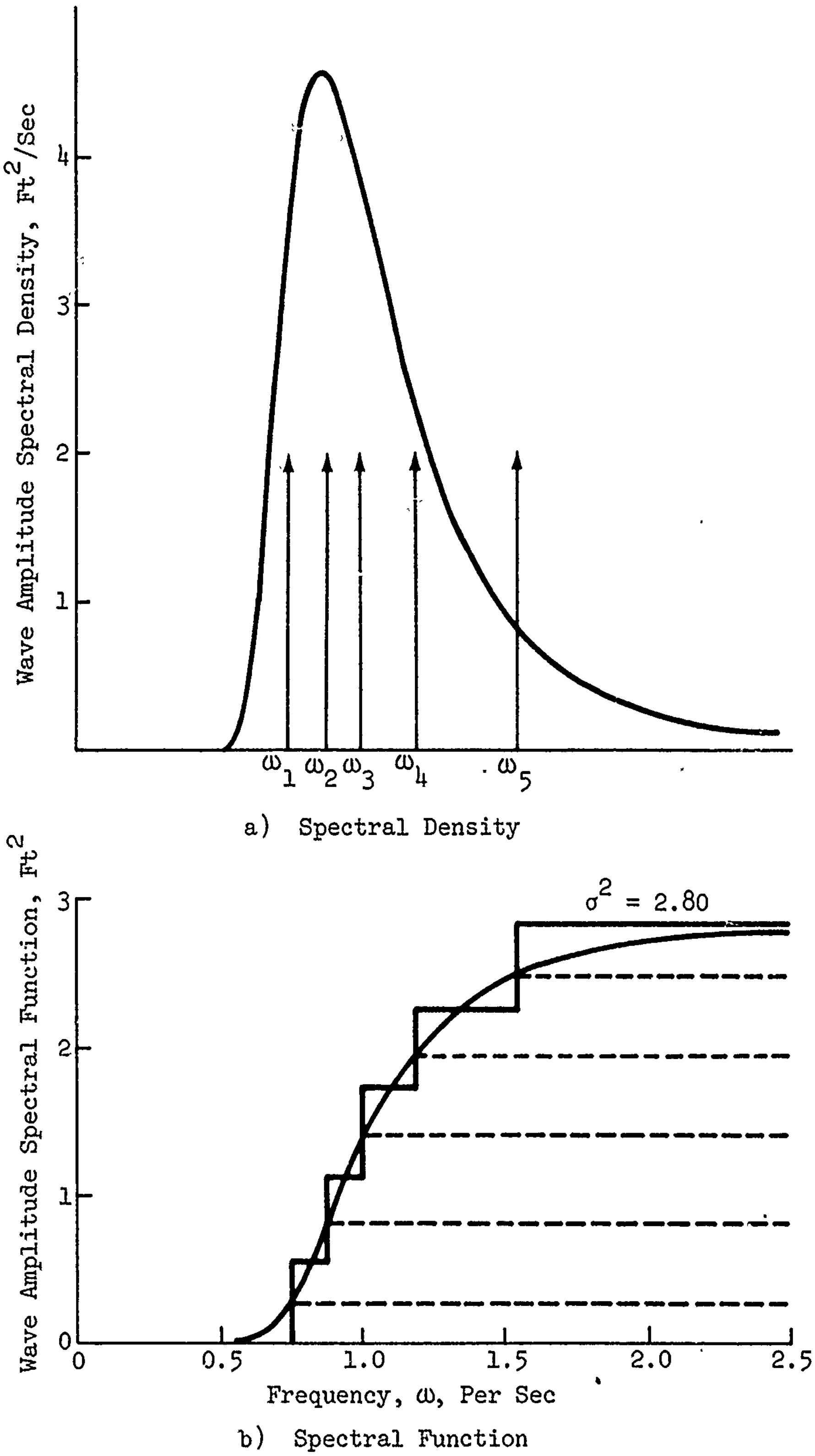

Fig. 2 19-Knot Pierson-Moskowitz Spectrum

$$\eta(t) = \sum_{i=1}^{N} [a_i \cos \omega_i t + b_i \sin \omega_i t]$$

where the a_i's and b_i's are identically distributed and uncorrelated Gaussian random variables with zero mean and variance, σ^2/N. This random process is stationary with zero mean and autocorrelation:

$$R(\tau) = (\sigma^2/N) \sum_{i=1}^{N} \cos (\omega_i \tau)$$

It is also ergodic in the mean, i.e., the time average

$$S = \frac{1}{2T} \int_{-T}^{T} \eta(t)\, dt$$

$$= \sum_{i=1}^{N} \frac{a_i \sin \omega_i T}{\omega_i T}$$

tends to the ensemble mean, zero, as T approaches infinity for all sample functions, i.e., independent of the random selection of the a_i's and b_i's. S has a Gaussian distribution with mean zero and variance, σ_S^2.

$$\sigma_S^2 = E\left\{\left[\sum_{i=1}^{N} \frac{a_i \sin \omega_i T}{\omega_i T}\right]^2\right\}$$

$$\sigma_S^2 = \frac{\sigma^2}{N} \sum_{i=1}^{N} \left[\frac{\sin \omega_i T}{\omega_i T}\right]^2$$

$$\sigma_S^2 < \left[\frac{2\sigma}{M\pi}\right]^2$$

for $T \geq M\pi/2\omega_{min}$, $M = 1, 3, 5, \ldots$. In particular for $T = M\pi/2\omega_{min}$, $\sigma_S^2 < (\sigma/T\omega_{min})^2$. For $N = 5$, $\omega_{min} = 0.75$, and $\sigma^2 = 2.80$. Taking $M = 5$, $T = 10.5$, and $\sigma_S^2 < 0.04515$ so that $\Pr\{|S| > 0.1\} < 0.027$.

Furthermore, in the limit as N gets large, this process is ergodic in variance. This can be shown by forming the time average of the square of $\eta(t)$:

$$W = \frac{1}{2T} \int_{-T}^{T} \eta^2(t)dt$$

$$W = \frac{1}{2T} \int_{-T}^{T} \left\{ \sum_{i=1}^{N} \left[a_i^2 \cos^2\omega_i t + b_i^2 \sin^2\omega_i t + 2a_i b_i \cos \omega_i t \sin \omega_i t\right]\right.$$

$$+ 2 \sum_{i=1}^{N} \sum_{\substack{j=1 \\ i \neq j}}^{N} \left[a_i a_j \cos \omega_i t \cos \omega_j t + b_i b_j \sin \omega_i t \sin \omega_j t\right.$$

$$\left.\left. + a_i b_j \cos \omega_i t \sin \omega_j t\right] \right\} dt$$

$$W = \frac{1}{2}\sum_{i=1}^{N}(a_i^2 + b_i^2) + \frac{1}{2}\sum_{i=1}^{N}(a_i^2 - b_i^2)\left(\frac{\sin 2\omega_i T}{2\omega_i T}\right)$$

$$+ \sum_{\substack{i=1 \\ i \neq j}}^{N}\sum_{j=1}^{N}\left[a_i a_j\left\{\frac{\sin(\omega_i - \omega_j)T}{(\omega_i - \omega_j)T} + \frac{\sin(\omega_i + \omega_j)T}{(\omega_i + \omega_j)T}\right\}\right.$$

$$\left. + b_i b_j\left\{\frac{\sin(\omega_i - \omega_j)T}{(\omega_i - \omega_j)T} - \frac{\sin(\omega_i + \omega_j)T}{(\omega_i + \omega_j)T}\right\}\right]$$

$$W_\infty = \lim_{T \to \infty} W = \frac{1}{2}\sum_{i=1}^{N}(a_i^2 + b_i^2)$$

If $W_\infty = \sigma^2$, the process would be ergodic in the variance. Note that based on the distribution of a_i, b_i, $\hat{W} = 2N/\sigma^2 \, W_\infty$ has the chi-square distribution with $2N$ degrees of freedom; therefore, the mean of W_∞ is σ^2 and its variance is σ^4/N. This proves that $\lim_{N \to \infty} W_\infty = \sigma^2$. Furthermore, the chi-square distribution supplies a quantitative estimate of the necessary order, N, to guarantee "practical" ergodicity in variance, given that the a_i's and b_i's are chosen completely at random. Suppose that a 25 percent error can be tolerated if it occurs no more than 35 percent of the time: $0.65 \approx \Pr\{0.75\,\sigma^2 < W_\infty < 1.25\,\sigma^2\} = \Pr\{1.5\,N < \hat{W} < 2.5\,N\}$. Thus $N = 15$ would suffice. Also, if N is taken to be 50, then the same 25 percent error would only occur 8 percent of the time, and a 10 percent error would occur with a 50/50 probability. Of course for computer simulation of random seas, a pseudo-random sampling algorithm can "stack the deck" by rejecting unfavorable sets of (a_i, b_i)'s.

To apply the random process to simulation of buoy motion, it is best to put the process $\eta(t)$ into state variable form. In the following, the great power of this method will be seen in its ability to represent exactly the distributed effects of a random sea. Water particle velocity is needed at each of 13 stations along the length of the buoy. In addition, the buoy experiences pitch, heave, and surge. What is needed are expressions for wave amplitude, $\eta_x(t)$, at an arbitrary horizontal inertial distance, x, from the reference and for vertical and horizontal components of orbital particle velocity, $W_{\eta_{xz}}(t)$ and $U_{\eta_{xz}}(t)$, respectively, at an arbitrary horizontal distance, x, from the reference and vertical distance, z, below the mean water surface.

Consider the auxiliary linear system

$$\dot{\overline{V}} = \Omega\overline{V} \quad ; \quad \overline{V}(0) = \overline{V}^o$$

where

$$\Omega = \begin{bmatrix} \Omega_1 & & & 0 \\ & \Omega_2 & & \\ & & \ddots & \\ 0 & & & \Omega_N \end{bmatrix}$$

and each Ω_i is

$$\Omega_i = \begin{bmatrix} 0 & \omega_i \\ -\omega_i & 0 \end{bmatrix}$$

If the initial condition vector, $\overline{V}^o$, has zero mean and covariance matrix $(\sigma^2/N)I$, then the random process for wave amplitude at the reference point is:

$$\eta(t) = \sum_{i=1}^{N} V_{2i-1}(t)$$

Recall that the relationship linking wavelength, λ, to frequency, ω, is

$$\lambda = 2g/\omega^2$$

where g is the acceleration of gravity. Therefore the wave amplitude at any x is

$$\eta_x(t) = \sum_{i=1}^{N} V^o_{2i-1} \cos\left(\omega_i t + \frac{2\pi x}{\lambda_i}\right) + V^o_{2i} \sin\left(\omega_i t + \frac{2\pi x}{\lambda_i}\right)$$

$$= \sum_{i=1}^{N} \cos\left(\frac{2\pi x}{\lambda_i}\right) V_{2i-1}(t) + \sin\left(\frac{2\pi x}{\lambda_i}\right) V_{2i}(t)$$

The vertical component of orbital velocity at any (x,z) is

$$W_{\eta_{xz}}(t) = \sum_{i=1}^{N} \exp\left(\frac{-2\pi z}{\lambda_i}\right)\omega_i\left[V^o_{2i-1} \sin\left(\omega_i t + \frac{2\pi x}{\lambda_i}\right) - V^o_{2i} \cos\left(\omega_i t + \frac{2\pi x}{\lambda_i}\right)\right]$$

$$= \sum_{i=1}^{N} \exp\left(\frac{-2\pi z}{\lambda_i}\right)\omega_i\left[\sin\left(\frac{2\pi x}{\lambda_i}\right) V_{2i-1}(t) - \cos\left(\frac{2\pi x}{\lambda_i}\right) V_{2i}(t)\right]$$

Similarly, the horizontal component is

$$U_{\eta_{xz}}(t) = \sum_{i=1}^{N} \exp\left(\frac{-2\pi z}{\lambda_i}\right)\omega_i\left[V^o_{2i-1} \cos\left(\omega_i t + \frac{2\pi x}{\lambda_i}\right) + V_{2i} \sin\left(\omega_i t + \frac{2\pi x}{\lambda_i}\right)\right]$$

$$= \sum_{i=1}^{N} \exp\left(\frac{-2\pi z}{\lambda_i}\right)\omega_i\left[- \cos\left(\frac{2\pi x}{\lambda_i}\right) V_{2i-1}(t) + \sin\left(\frac{2\pi x}{\lambda_i}\right) V_{2i}(t)\right]$$

For the sake of discussion, suppose that 20 z-locations and 5 x-locations were required. Then the above three equations would define an output vector with 205 components, i.e.,

$$\overline{W}^T = \left[\eta_{x_1}, \eta_{x_2}, \ldots, \eta_{x_5}, W_{\eta_{x_1 z_1}}, W_{\eta_{x_1 z_2}}, \ldots, W_{\eta_{x_5 z_{20}}}, U_{\eta_{x_1 z_1}}, \ldots, U_{\eta_{x_5 z_{20}}}\right]$$

$$\overline{W} = H\overline{V}$$

where H is a 205 by $2N$ matrix of constants.

For the case in which the buoy is considered to satisfy linear small perturbation differential equations (10 in our current model), the situation is vastly simplified. Only the stationarity of the random process needs to be utilized. The covariance matrix of buoy state variables is time invariant, and these variables have zero means and obey a multivariate Gaussian probability distribution. Thus, the probability distribution is completely determined by the steady-state covariance matrix, which is computed as the routine solution of an algebraic system of equations. There is no need for either random number generation or numerical integration. The computer code for this computation exists and is well tested.

On the other hand, this method of representing the random sea is also well suited to the nonlinear case. Naturally, there are

complications in the computations. Care must be exercised in selecting a random sample for the set $(a_i, b_i, i = 1, 2, \ldots, N)$, which corresponds to a particular observed sea. Large enough N and T must be employed. The simulation technique is of the Monte Carlo type in which ergodicity of the mean and variance of both the driving process and the buoy variables are assumed. Even with these assumptions, the use of a finite simulation time interval implies a certain level of error in the estimation of the mean vector and the covariance matrix. The means of the buoy variables are not necessarily zero, and their multivariate distribution is not Gaussian and, therefore, not describable by a finite number of statistical moments. Nevertheless, useful conclusions can still be drawn about the statistics of buoy response through time averages of the response and the squared response to any particular "sample wave" generated by the above technique.

In conclusion, it is expected that use will be made of both linear models of buoy dynamics for which precise statistical conclusions can be drawn, and nonlinear models, which account especially for both the limited excess buoyancy and the inability of a cable to offer compressive resistance.

References

1. Rossi, M., "Optimal Design of an Automatic Control System for Submerged Hydrofoil Boats Operating in a Random Seaway," IEEE Conf. Eng'g. in the Ocean Environment, Newport, Rhode Island, September 13-15, 1972.

2. Papoulis, A., Probability, Random Variables, and Stochastic Processes, McGraw-Hill, 1965.

3. Gikhman, I. I. and Skorokhod, A. V., Introduction to the Theory of Random Processes, W. B. Saunder, 1969 (Russian, Nauka Press, Moscow, 1965).

4. Neumann, G. and Pierson, Jr., W. J., Principles of Physical Oceanography, Prentice-Hall, 1966.

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